What has my network learned? The rainbow model of deep networks

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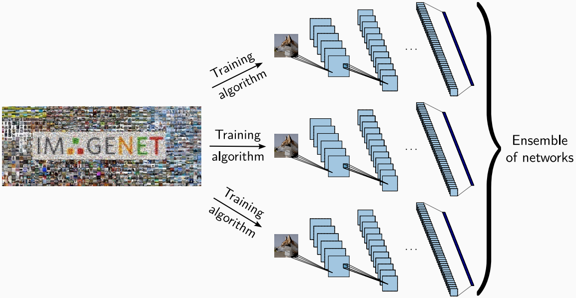
What has the network learned?

Every time we train a network, we get a different set of weights because of the random initialization



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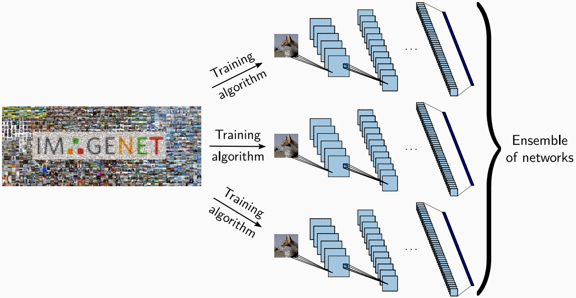

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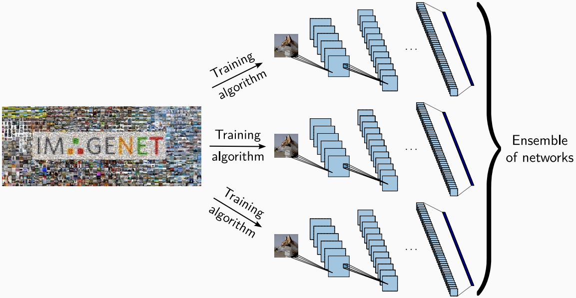
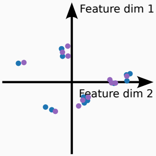


- What is random and what is stable across different training runs?
- **What is the distribution of trained network weights?**

Comparing first layer weights



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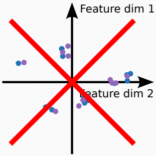

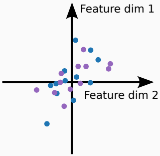

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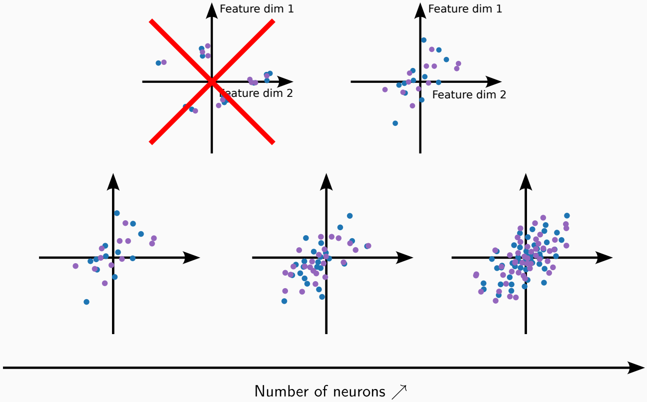

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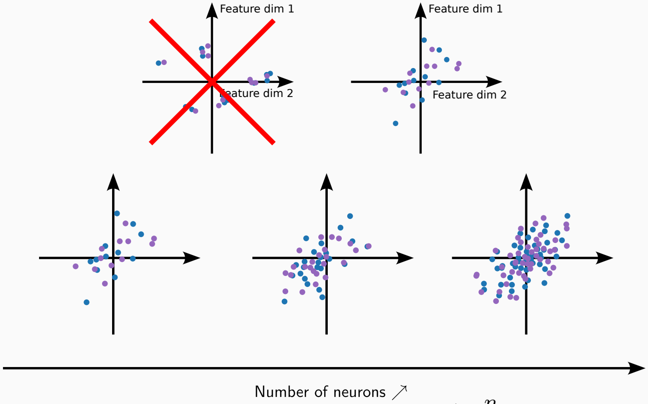

Mean-field (infinite-width) limit

$$\frac{1}{n}\sum_{i=1}^{n}\delta_{w_i} \xrightarrow[n\to\infty]{} \pi$$

(Chizat and Bach, 2018; Mei et al., 2018; Rotskoff and Vanden-Eijnden, 2018; Sirignano and Spiliopoulos, 2020)

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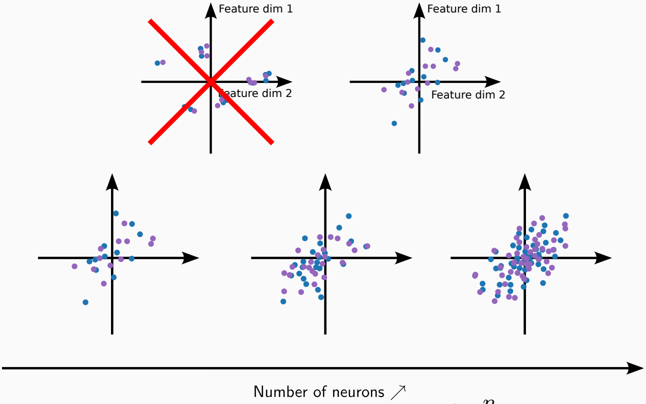

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Neurons are random samples from some fixed distribution
(random features)

Comparing first layer activations

Random feature activations: $\phi(x) = n^{-1/2}(\sigma(\langle w_i, x\rangle))_{i\leq n}$ with $w_i \sim \pi$ i.i.d.

$$\langle \phi(x), \phi(x') \rangle = \frac{1}{n}\sum_{i=1}^{n} \sigma(\langle w_i, x\rangle)\,\sigma(\langle w_i, x'\rangle) \to \mathbb{E}_{w\sim\pi}\Big[\sigma(\langle w, x\rangle)\,\sigma(\langle w, x'\rangle)\Big]$$

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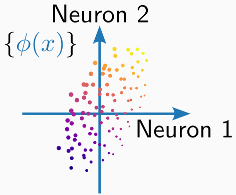

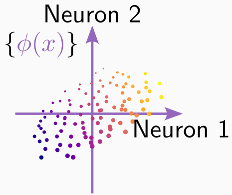

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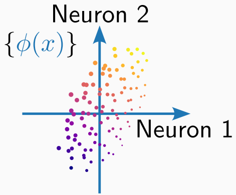
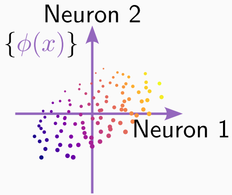
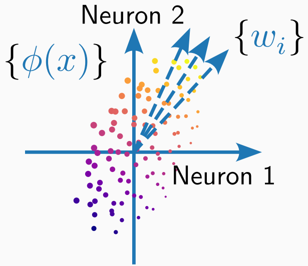


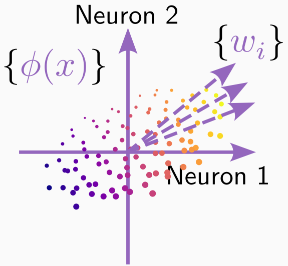


Activations are equal up to rotations: they correspond to a deterministic representation expressed in a random basis

(Rahimi and Recht, 2007; Haxby et al., 2011; Kornblith et al., 2019)

Comparing second layer weights





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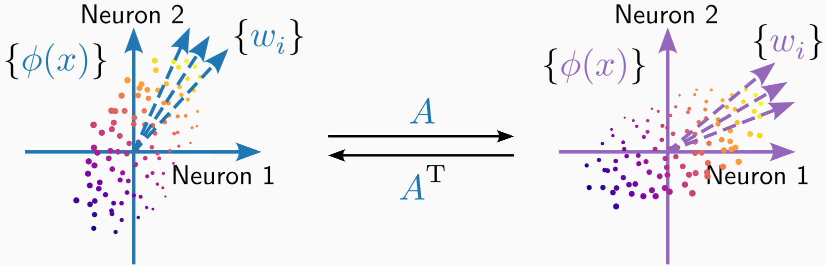

$$\min_{A^{\mathrm{T}}A=\mathrm{Id}} \; \mathbb{E}_x\left[\|A\,\phi(x)-\phi(x)\|^2\right] \qquad A\,\phi(x)\approx\phi(x)$$

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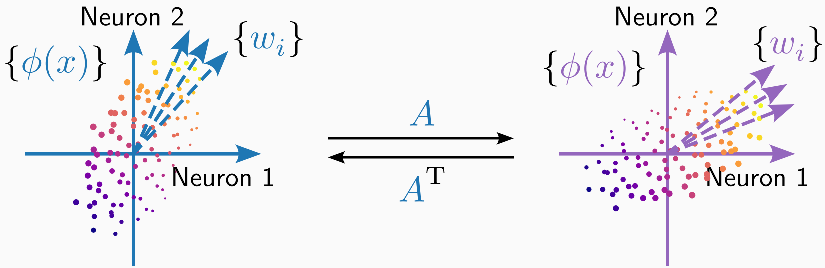

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$$\langle w_i, \phi(x)\rangle \approx \langle w_i, A^{\mathrm{T}}\phi(x)\rangle = \langle A\,w_i, \phi(x)\rangle$$

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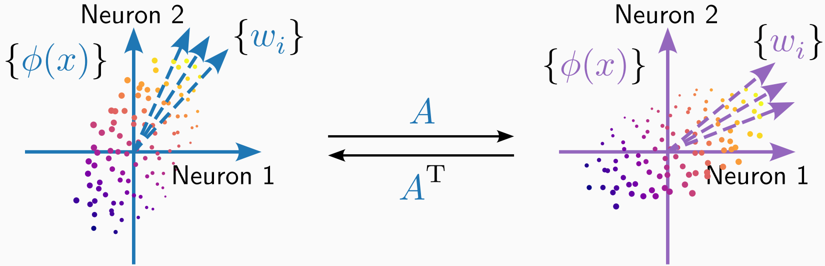

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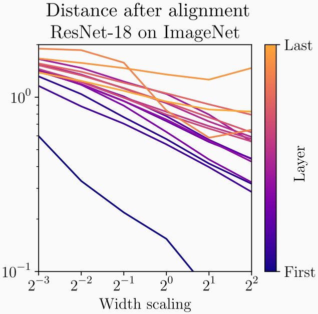
Neurons are random samples from some fixed distribution

expressed in the random basis of its input activations

Comparing aligned activations and weights

Comparison between activations



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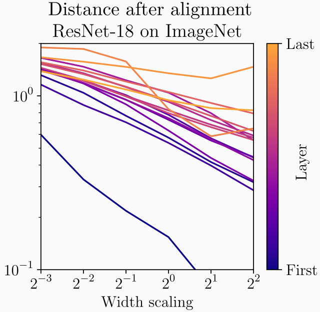

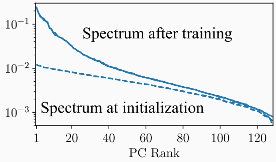

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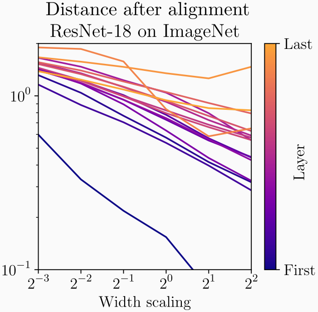

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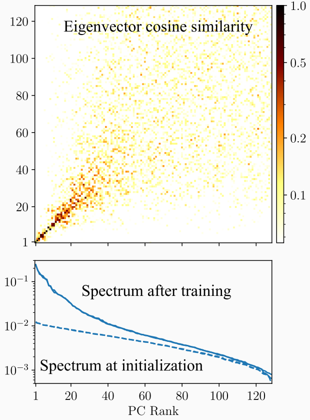

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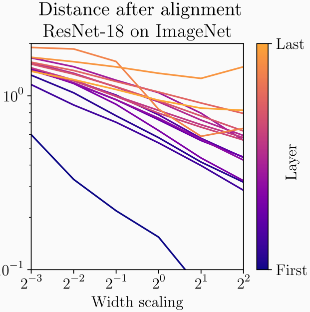

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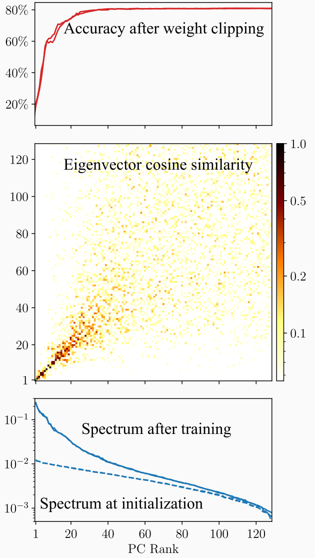
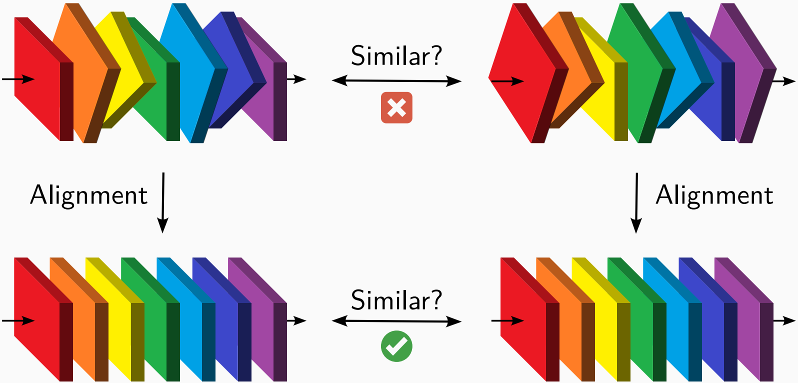


Summary



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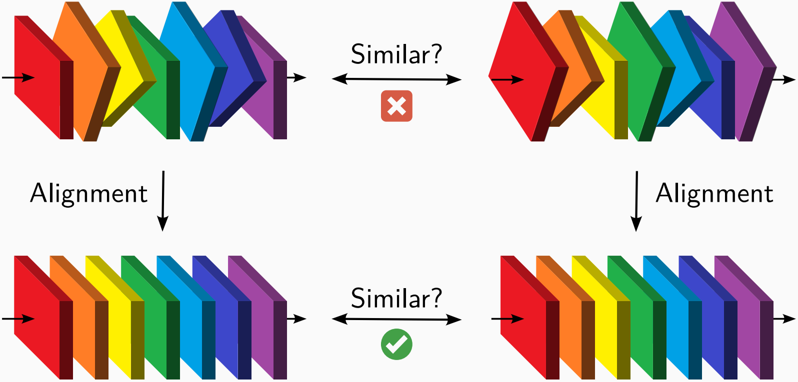
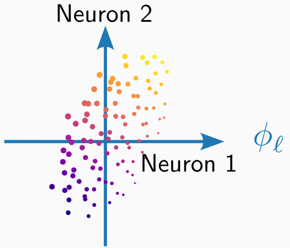


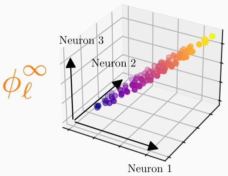
In the infinite-width limit, there is a unique deterministic network and finite-width networks can be seen as random feature discretizations of it.

The rainbow model of trained network weights

Model parameters: weight distributions π_ℓ and representations ϕ_ℓ^∞

Iterative sampling procedure: assume ϕ_ℓ has been defined





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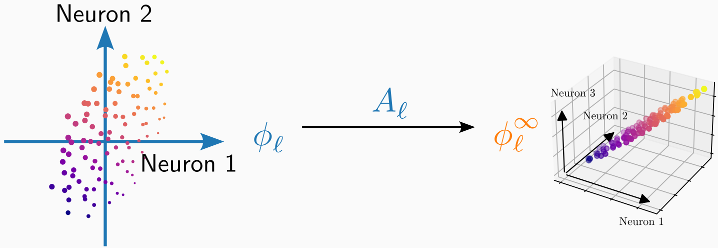

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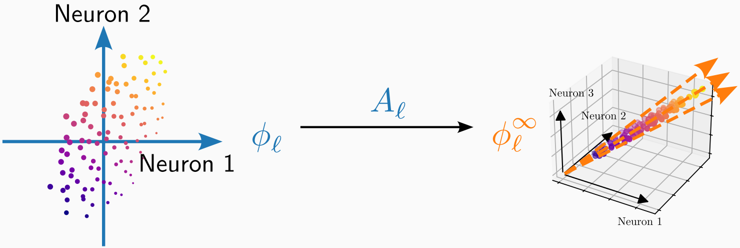

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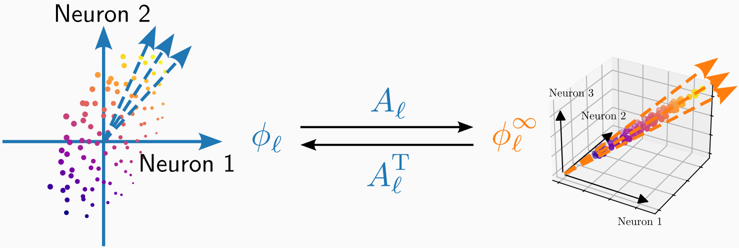

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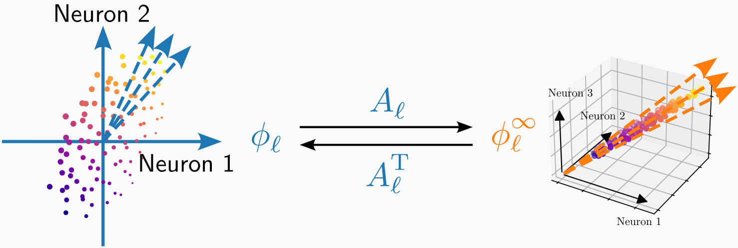

Theorem: $\forall \ell, A_\ell \, \phi_\ell \to \phi_\ell^\infty$ *polynomially in the widths.*

Assumptions: π_ℓ *has finite fourth-order moments + capacity conditions at each layer.*

A simpler model: Gaussian rainbow networks

Model fully specified by weight covariances C_ℓ at each layer

- Sample $w_{1,i} \sim \mathcal{N}(0, C_1)$
- Compute A_1 by aligning ϕ_1 to ϕ_1^∞
- Sample $w_{2,i} \sim \mathcal{N}\left(0, A_1^{\mathrm{T}} C_2 A_1\right)$
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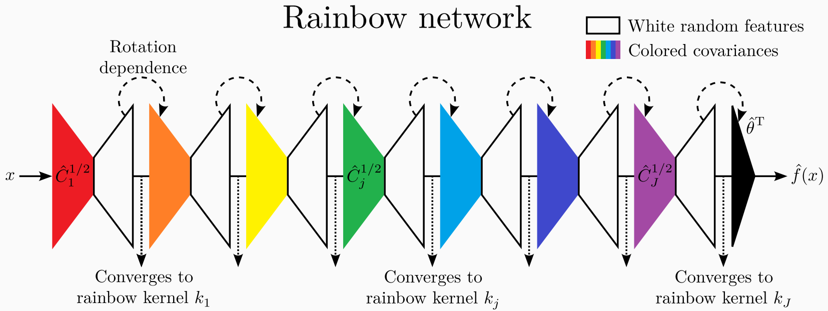

Evaluating the accuracy of rainbow networks

- Train a scattering network on CIFAR-10 (**fixed spatial filters** + learned channel weights) (Guth, Zarka, and Mallat, 2022)
- Extract channel covariances C_ℓ at each layer
- Generate random weights with the same **aligned** covariances
- Evaluate accuracy on test set!

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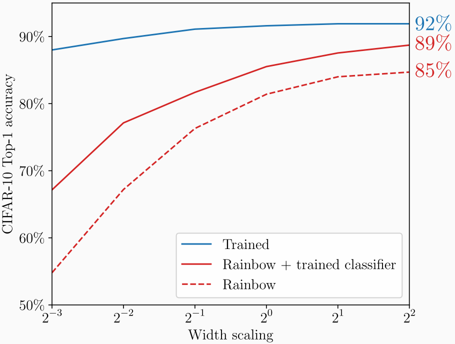
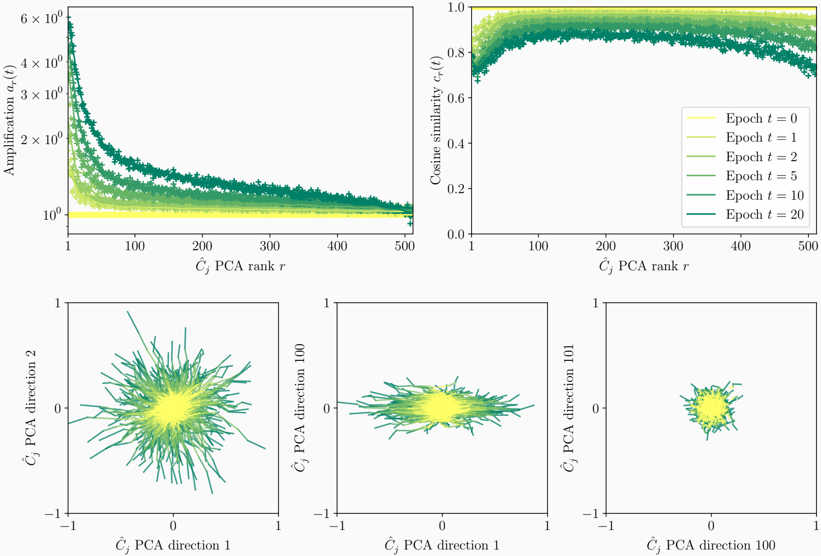


Training dynamics



Conclusion

- What has been learned? Weight distributions, sometimes just covariances
- How do they depend on the training data?
- Trained networks (and real-world datasets) as objects of scientific study
- Opens many questions in optimization (regime of validity of the model?) and generalization (properties of rainbow kernels?)

https://arxiv.org/abs/2305.18512

https://bonnerlab.github.io/ccn-tutorial/pages/analyzing_neural_networks.html