Towards a Mathematical Understanding of Deep Convolutional Neural Networks

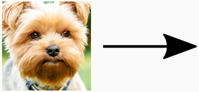
Florentin Guth

Learning from data

Image classification

“cat”

“dog”



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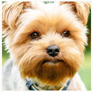

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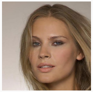

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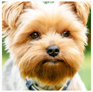

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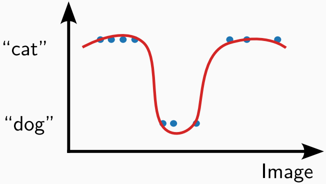

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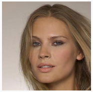

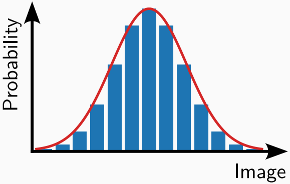

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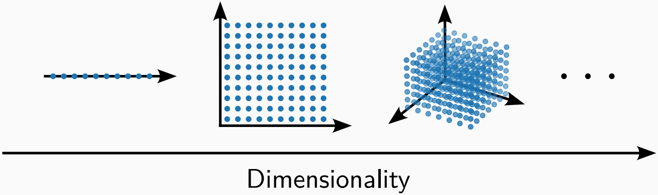

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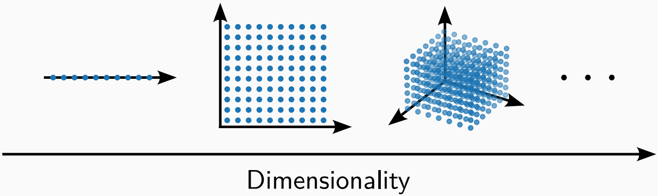
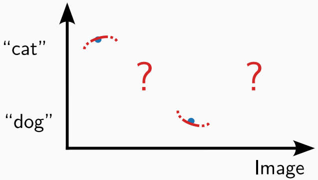
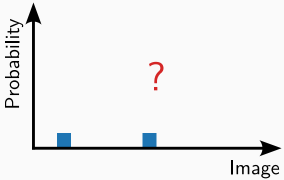
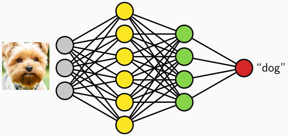


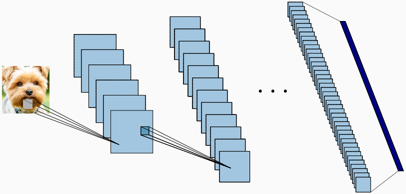
How to learn in high dimensions?





Enter deep learning





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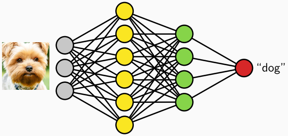

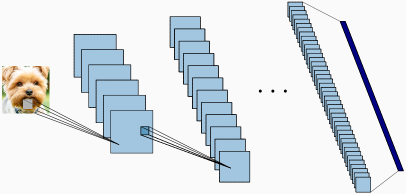

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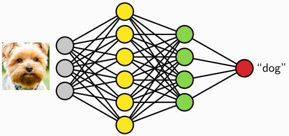

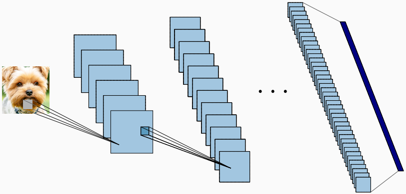

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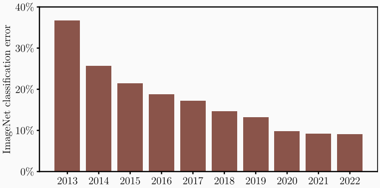

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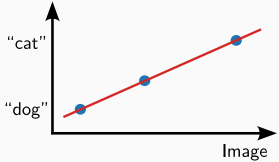

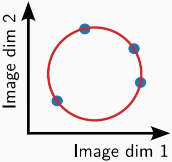

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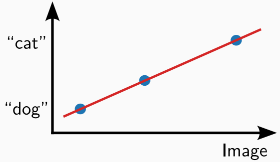

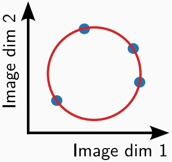

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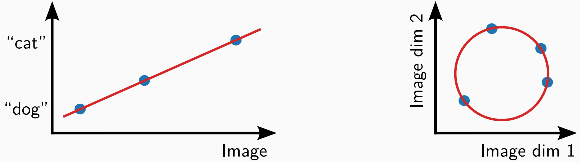

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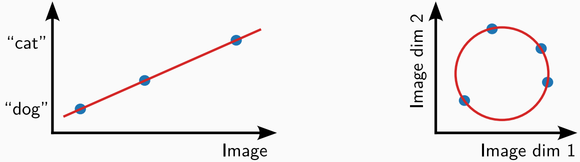

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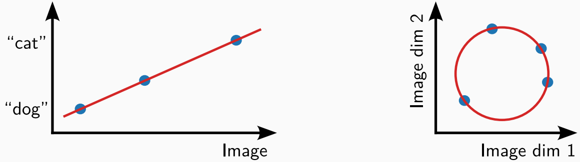

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- In the data distribution: what are its properties?
- In the network computations: what are its functional blocks?
- In the network weights: what has been learned?

Outline

Exploiting Structure in Image Probability Distributions

Enforcing Structure in Convolutional Network Architectures

Discovering Structure in Learned Network Weights

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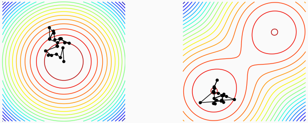

Multiscale generation

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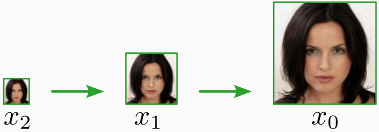

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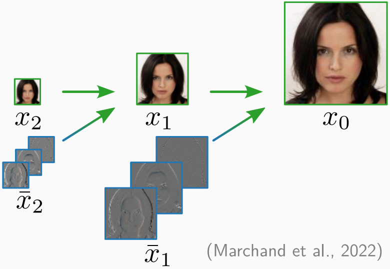

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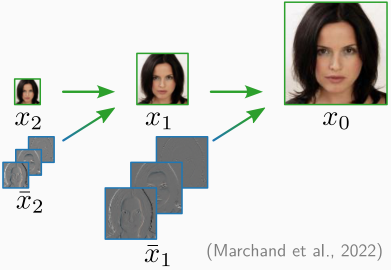

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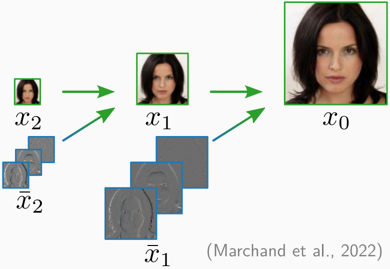

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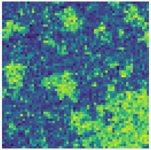
What are the properties of these *conditional* distributions?

Conditional locality

A “simpler” class of image distributions: physical fields



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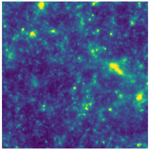

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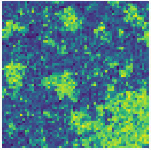

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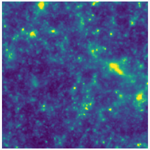

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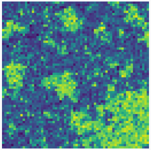

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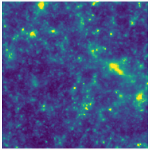

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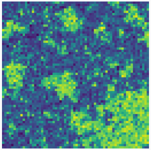

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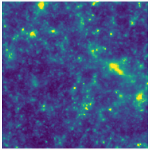

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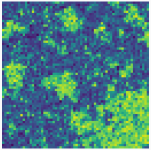

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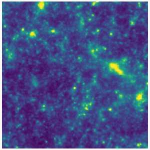

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- $E(\bar{x}_j | x_j)$ then decomposes as a sum of local potentials (conditional Markov random field)

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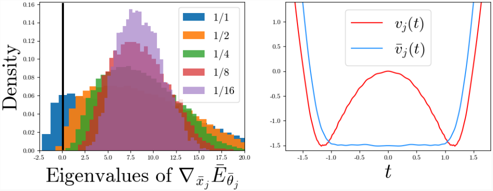

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G*, Lempereur*, Bruna, and Mallat. Conditionally strongly log-concave generative models. *ICML*, 2023.

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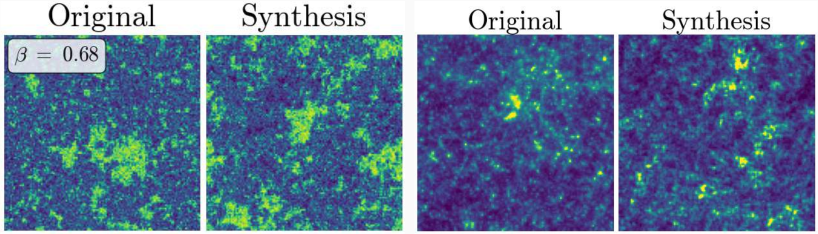

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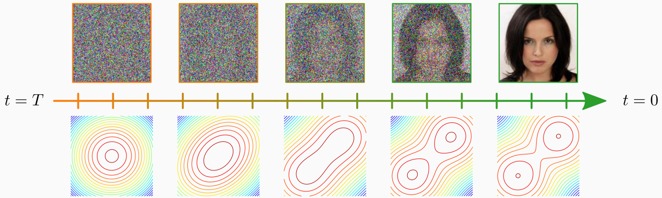

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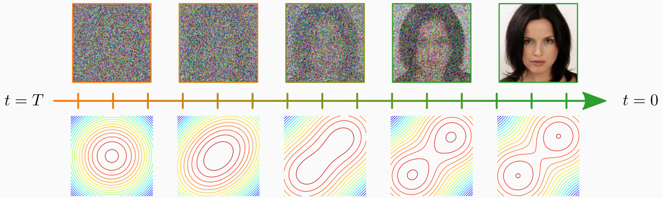
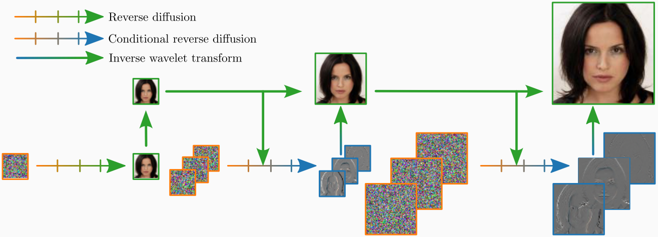


Diffusion models solve the issues associated with non-log-concavity (Song et al., 2021; Chen et al., 2022). Remaining burning question: how do deep networks learn the score?

Conditionally local diffusion models

Benefits of combining diffusion models with multiscale approaches?



G, Coste, De Bortoli, and Mallat. Wavelet score-based generative modeling. *NeurIPS*, 2022.
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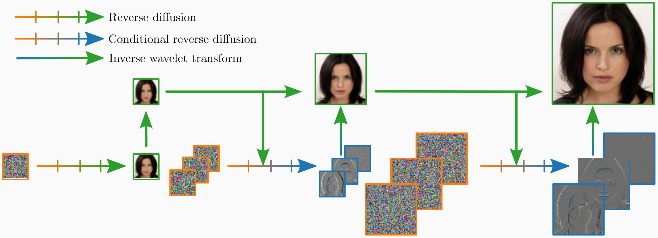
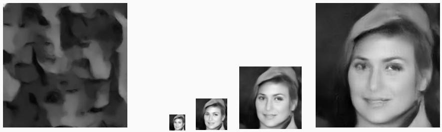


Locality:



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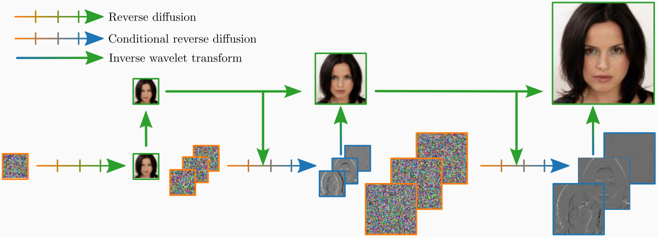

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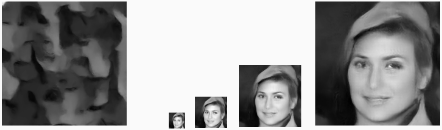

Sampling efficiency:

G, Coste, De Bortoli, and Mallat. Wavelet score-based generative modeling. *NeurIPS*, 2022.
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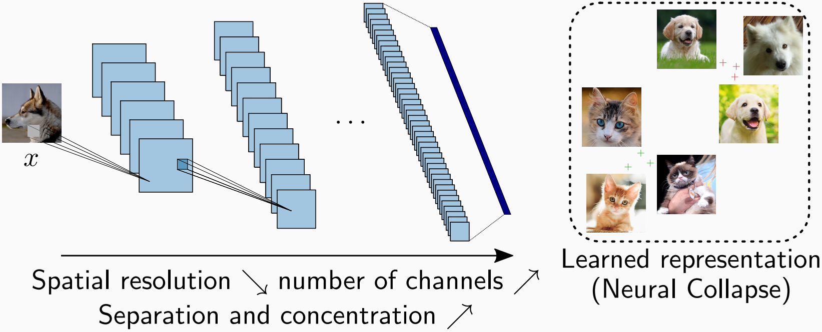
Outline

Neural collapse

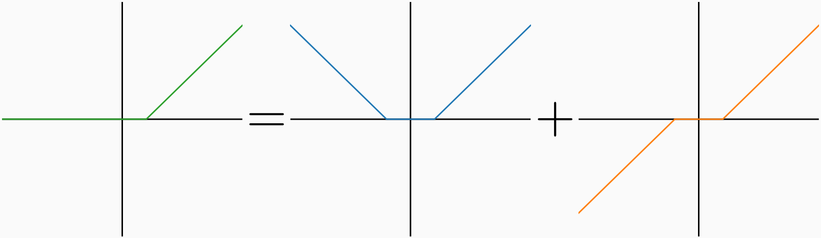


CNN classifiers simultaneously move spatial information into channels and increase linear separation

Can we define a non-linear operator with these properties?

Decomposition of ReLU

ReLUs can can be separated in two opposite non-linearities with an even-odd decomposition:



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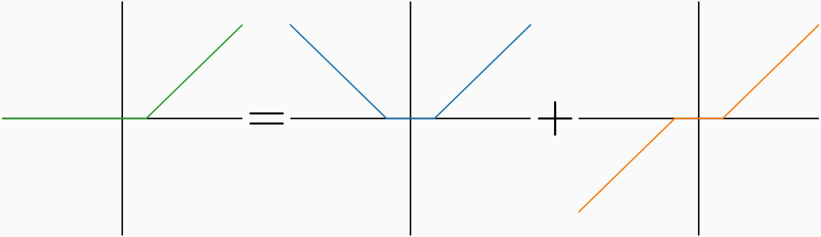

- Absolute value: collapses the sign, preserves the amplitude

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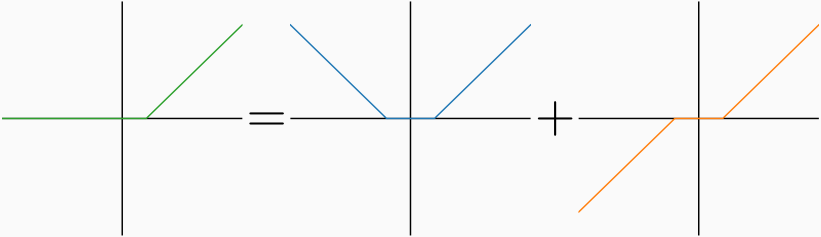
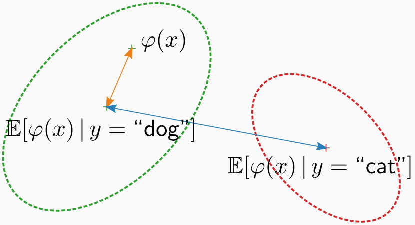


- Absolute value: collapses the sign, preserves the amplitude
- Soft-thresholding: preserves the sign, thresholds the amplitude

Concentration with soft-thresholding



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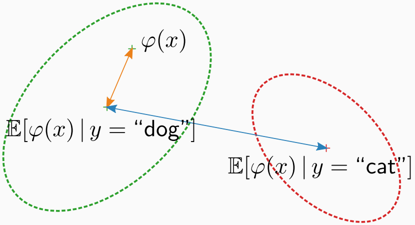

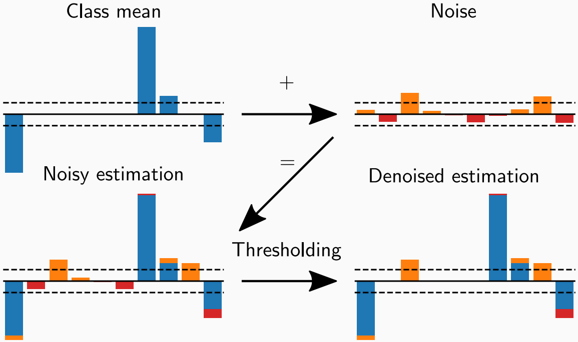

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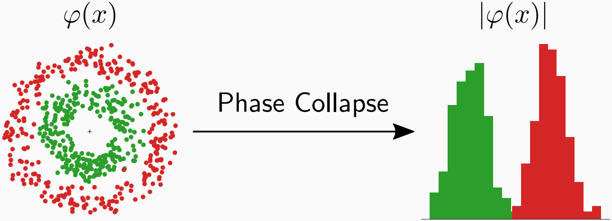
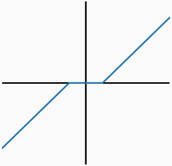
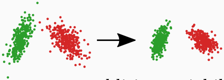


Comparison between sparsity and phase collapse

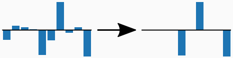
Concentration with soft-thresholding



Odd part of ReLU
Collapses small amplitudes

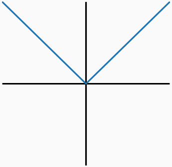


Concentrates additive variability
Does not separate class means

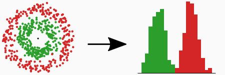


Performs denoising
Cannot be further sparsified

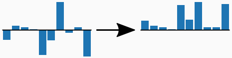
Separation with complex modulus



Even part of ReLU
Collapses complex phases

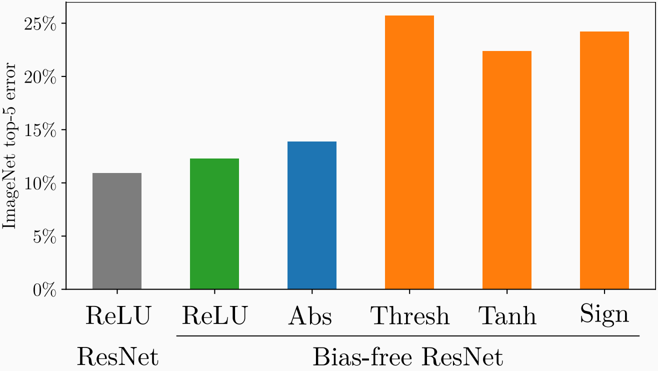


Concentrates multiplicative variability
Separates class means



Computes support
Can be further sparsified

Phase collapse versus sparsity: numerical results



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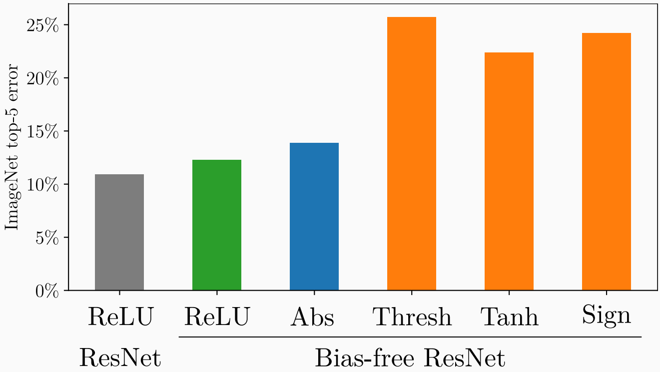

Phase collapse is sufficient to achieve good performance, while any non-linearity which preserves the phase is not. Phase collapse is thus also necessary.

How far can we further constrain the network?

Diagonalizing local translations

Known source of within-class variability: local translations

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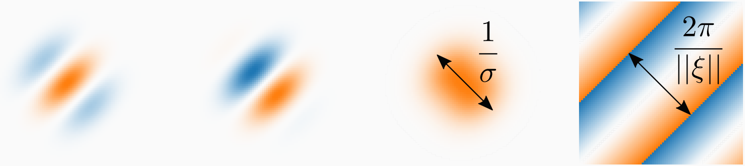


$\mathrm{Re}(\psi)$ $\mathrm{Im}(\psi)$ $|\psi|$ $\alpha(\psi)$

Small translations τ of an image x become phase shifts:

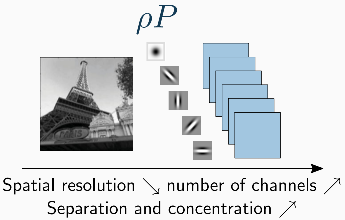
$$(\tau \cdot x) * \psi \approx e^{-i\xi \cdot \tau}(x * \psi)$$

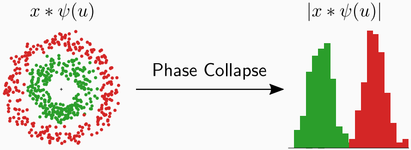
with a relative error bounded by $\sigma|\tau|$: approximate diagonalization!

The phase collapse operator

Constrain the spatial filters with the phase collapse operator:

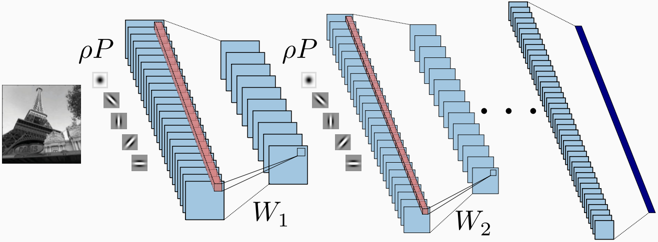
$$\rho P x(u) = \Big(x * \phi(2u), \left(|x * \psi_\theta(2u)|\right)_\theta \Big)$$





- Mathematical definition: no learning
- Combines linear and non-linear invariants to local translations
- All the desired properties!
- What accuracy can we achieve with this?

Learned scattering network



- Simplified architecture with phase collapses and minimal learning
- No learned spatial filters nor biases
- Only one learned component: channel matrices at every layer
- Reaches ResNet-18 accuracy with only 11 layers

Zarka, G, and Mallat. Separation and concentration in deep networks. *ICLR*, 2021.

G, Zarka, and Mallat. Phase collapse in neural networks. *ICLR*, 2022.

Outline

What has the network learned?

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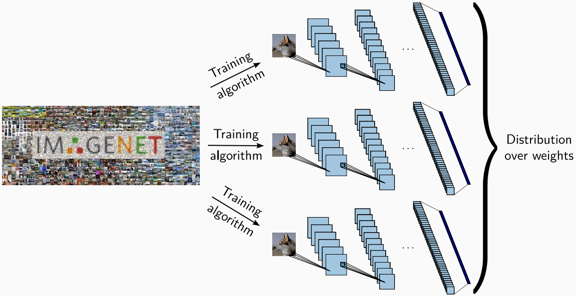

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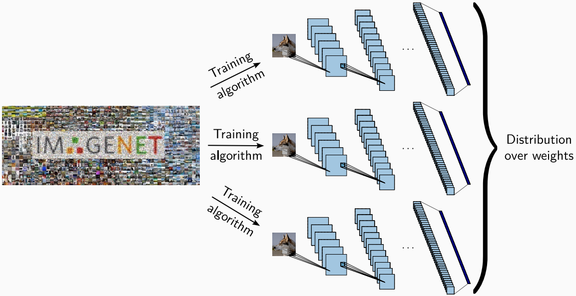

What is the distribution of trained network weights?

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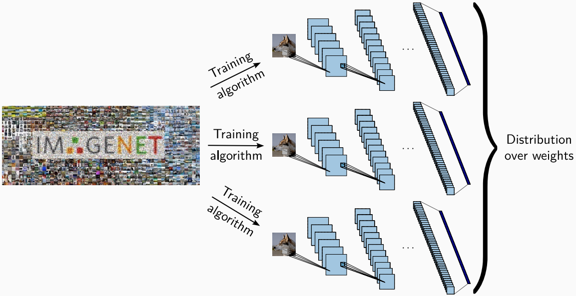
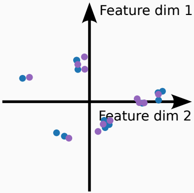


What is the distribution of trained network weights?

- Many parameters: laws of large numbers

Law of large numbers 1: weight statistics



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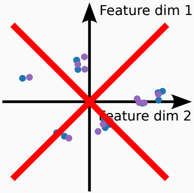

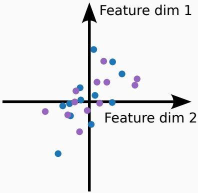

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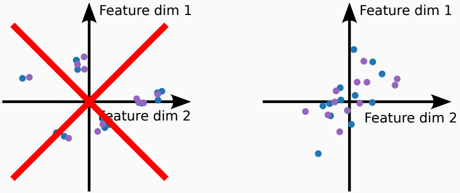
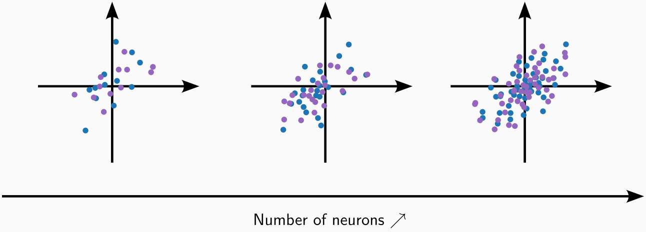


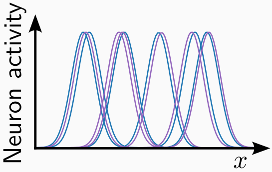
First law of large numbers: statistics of the neuron weights



Mean-field (infinite-width) limit of neural networks

(Chizat and Bach, 2018; Mei et al., 2018; Rotskoff and Vanden-Eijnden, 2018; Sirignano and Spiliopoulos, 2020)

Law of large numbers 2: representation geometry



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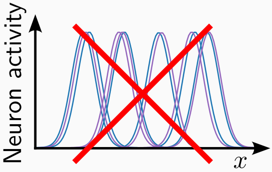

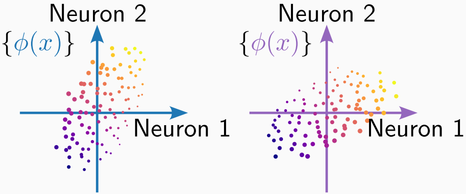

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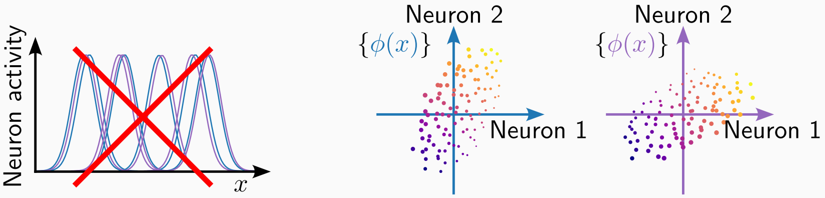

Second law of large numbers: geometry of the representation

(Rahimi and Recht, 2007)

$$\langle \phi(x), \phi(x') \rangle = \frac{1}{n} \sum_{i=1}^{n} \rho(\langle w_i, x \rangle)\, \rho(\langle w_i, x' \rangle) \to \mathbb{E}_{w \sim \pi}\Big[\rho(\langle w, x \rangle)\, \rho(\langle w, x' \rangle)\Big]$$

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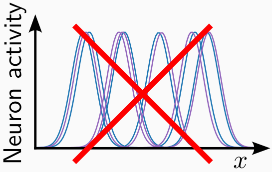

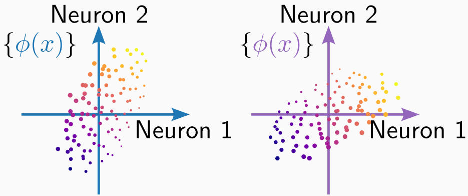

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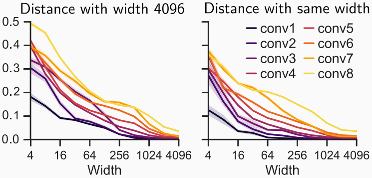
$$\mathbb{E}_{x,x'}\Big[\big(\langle \phi(x), \phi(x') \rangle - \langle \phi(x), \phi(x') \rangle\big)^2\Big]$$

(Kornblith et al., 2019)



Network alignment

$$\langle \phi(x), \phi(x') \rangle$$
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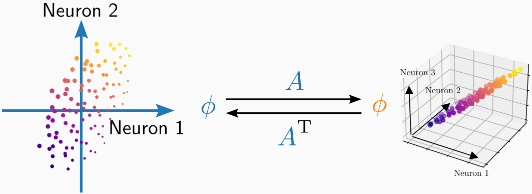

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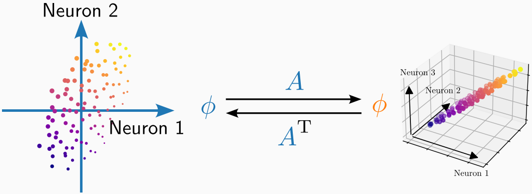

Define the alignment A with $\min_{A^{\mathrm{T}}A=\mathrm{Id}} \; \mathbb{E}_x\left[\|A\,\phi(x) - \phi(x)\|^2\right]$

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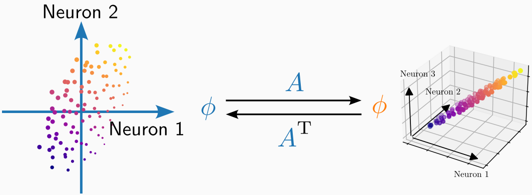

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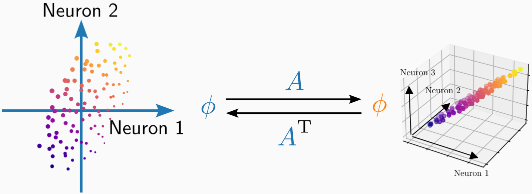

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Theorem: *If neuron weights are i.i.d. samples from π, then $A\,\phi \to \phi$ in mean square, polynomially in the width, and independently of the dimension.*

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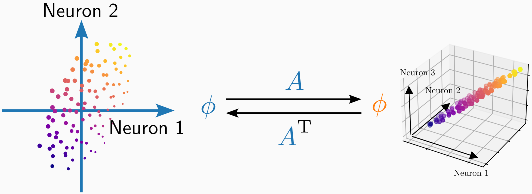

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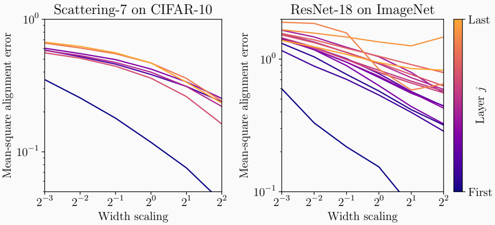
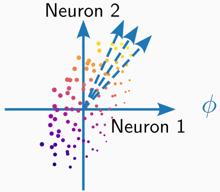
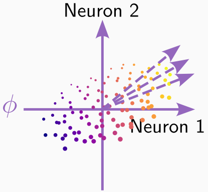


Mean-field limit in hidden layers





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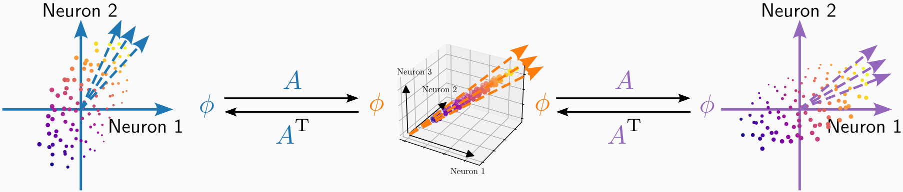

$$\langle w, \phi(x) \rangle = \langle Aw, A\phi(x) \rangle \approx \langle Aw, \phi(x) \rangle$$

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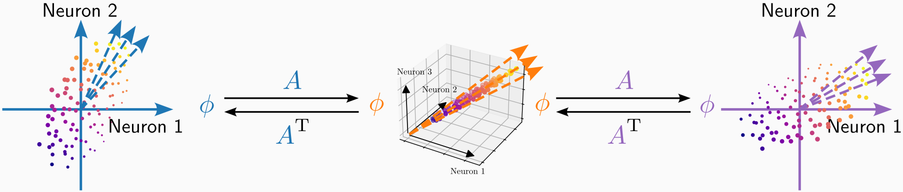

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In deeper layers, we expect a mean-field limit on the **aligned** neuron weights $\{Aw\}$

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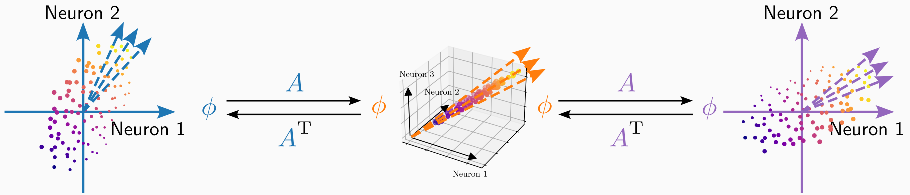

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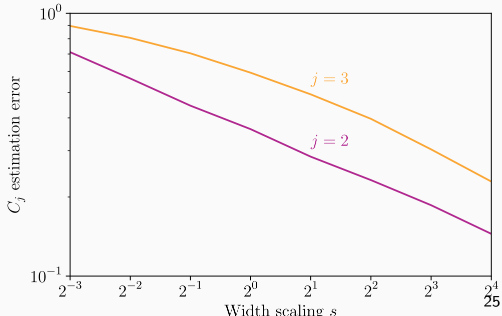

A probabilistic model of network weights

"Maximum-entropy" model under the mean-field constraints at each layer

A probabilistic model of network weights

"Maximum-entropy" model under the mean-field constraints at each layer

Generative model of network weights

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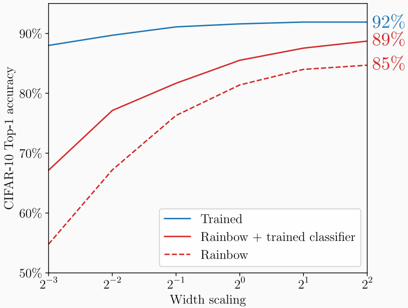
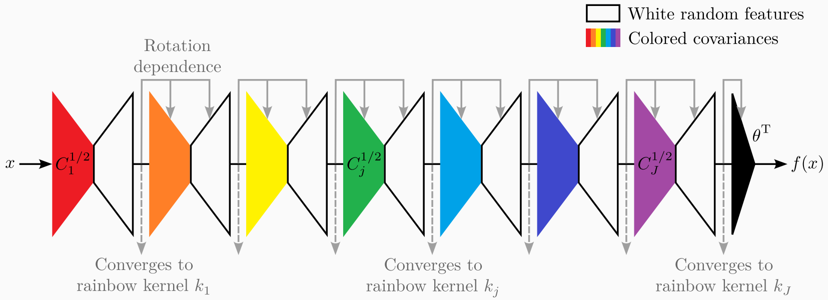


Covariance and dimensionality: the rainbow model



G, Ménard, Rochette, and Mallat. A rainbow in deep network black boxes. arXiv, 2023.

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Further research:

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Thank you!